MEASURABLE FUNCTION

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- Why we need measurable functions ?
- We ultimately intend to define an integration process modeled on Riemann integration which should be stronger than Riemann integration.
- Functions are the objects which we integrate,
- We need to define a special class of functions which we will like to integrate.

Definition 1.

• Let $f : D \to \mathbb{R}$ be a real valued function defined on a measurable subset D of $\mathbb{R}$.
• Then f is called Lebesgue measurable (more briefly, measurable) if one of the following holds.

(i) $\{x \in D : \; f(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

(ii) $\{x \in D : \; f(x) \leq \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

(iii) $\{x \in D : \; f(x) < \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

(iv) $\{x \in D : \; f(x) \geq \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$.

- Consider the following simple observations:
- $(\alpha, \infty)^c = (-\infty, \alpha]$.
- $(-\infty, \alpha)^c = [\alpha, \infty)$.
- $(\alpha, \infty) = \bigcup_{n=1}^{\infty} [\alpha + \frac{1}{n}, \infty)$.
- $[\alpha, \infty) = \bigcap_{n=1}^{\infty} (\alpha - \frac{1}{n}, \infty)$.
- Measurable sets form a σ-algebra,

- We could have considered extended real valued functions f,
- In addition to above mentioned properties, $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ must also be measurable.

Theorem 1.

If f, g are measurable functions then so are cf, where $c \in \mathbb{R}$, $f+g$, $f-g$, f^2, $|f|$, $\min\{f,g\}$, $\max\{f,g\}$.

(i) If $c = 0$ then the result is obvious. Let $\alpha \in \mathbb{R}$ be given.

- If $c > 0$
- $\{x \in D : cf(x) > \alpha\} = \{x \in D : f(x) > \frac{\alpha}{c}\}$.
- If $c < 0$
- $\{x \in D : cf(x) > \alpha\} = \{x \in D : f(x) < \frac{\alpha}{c}\}$.

(iii) By (i) $-g = (-1)g$ is measurable as g is measurable

- f is measurable $\Rightarrow f - g = f + (-g)$ is also measurable by (ii).

(iv) Let $\alpha \in \mathbb{R}$ be given.

- $\{x \in D : f^2(x) \geq \alpha\} = D$ if $\alpha \leq 0$
- $\{x \in D : f^2(x) \geq \alpha\} =$

$= \{x \in D : f(x) \geq \sqrt{\alpha}\} \cup \{x \in D : f(x) \leq -\sqrt{\alpha}\}$ if $\alpha > 0$.

$\Rightarrow f^2$ is also measurable.

(v) That fg is measurable follows from the above results and the fact that

- $fg = \frac{(f+g)^2-(f-g)^2}{4}$.

Theorem 2.

- If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions then
- $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ and $\lim_n f_n$ (if exists) are all measurable.

- (i) $\{x \in D : \sup_n f_n(x) > \alpha\} =$
$$= \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) > \alpha\}.$$
- Since $\{x \in D : f_n(x) > \alpha\}$ is measurable for each n and countable union of measurable sets is measurable, so $\{x \in D : \sup_n f_n(x) > \alpha\}$ is measurable.
- $\sup_n f_n$ is measurable.

- (ii) The result follows from the observation that
- $\{x \in D : \inf_n f_n(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x \in D : f_n(x) < \alpha\}.$

Let $f : D \to \mathbb{R}$ be a measurable function and let $A = \{x \in D : f(x) = 0\}$. If $\frac{1}{f}$ is defined to be α on A for some $\alpha \in \mathbb{R}$ then $\frac{1}{f}$ is measurable. If $\mu(A) = 0$ then $\frac{1}{f}$ is measurable irrespective of what values we assign to it on A.

- $D - A$ is measurable
- $f : D - A \to \mathbb{R}$ is measurable
- $\frac{1}{f}$ is well defined on $D - A$ and is also measurable.

- For any real number c

$$\begin{aligned} \{x \in D : \frac{1}{f(x)} > c\} &= \{x \in D - A : \frac{1}{f(x)} > c\} \text{ if } c \geq \alpha, \\ &= \{x \in D - A : \frac{1}{f(x)} > c\} \cup A \text{ if } c < \alpha. \end{aligned}$$

- $\frac{1}{f}$ is measurable.

- Define $g : D \to \mathbb{R}$ by $g(x) = f(x)$ when $f(x) \neq 0$
- $g(x) = \beta$ for all $x \in A$ where $\beta \neq 0$.
- $g = f$ almost everywhere and so g is measurable.

- Since $g \neq 0$ on D, so $\frac{1}{g}$ is also measurable.
- $\frac{1}{g} = \frac{1}{f}$ almost everywhere and so $\frac{1}{f}$ is measurable irrespective of the values we assign to $\frac{1}{f}$ on A.

Thank you!