TOPOLOGY

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Definition

Let $A, B \subset_{\text{closed}} X$ and $A \cap B = \emptyset$.
The sets A and B are separated by a function if there exists a continuous function

$$f : X \to [0, 1]$$

such that

1. $f(a) = 0, \forall a \in A$, i.e., $f(A) = \{0\}$ and
2. $f(b) = 1, \forall b \in B$, i.e., $f(B) = \{1\}$.

Question

When every two disjoint closed subsets in a topological space can be separated by a continuous function?

Answer: Urysohn's Lemma: Characterizes topological spaces where every two disjoint closed subsets can be separated by a continuous function.

Definition (Urysohn function)

Let X be a topological space and A, B be two disjoint closed subsets of X.
In this setting:
A continuous function $f : X \to [0, 1]$ is called Urysohn function if

$$f(A) = \{0\} \text{ and } f(B) = \{1\}$$

Implications

Urysohn's lemma is key in the proof of many other theorems, for instance

- Tietze extension theorem
- Paracompact Hausdorff spaces equivalently admit subordinate partitions of unity

Theorem (Urysohn's Lemma)

Let X be a normal space and $A, B \subset_{closed} X$ with $A \cap B = \emptyset$. Then there exists an Urysohn function separating A and B.
Moreover, the converse is also true.

Proof of the converse

Converse is easy. So, let us prove the converse first:

- Assume, for any disjoint closed sets A and B there exists an Urysohn function $f : X \to [0, 1]$ separating them.
- To show: there are disjoint open sets U, V s.t.

$$A \subset U \text{ and } B \subset V.$$

- Consider $U = f^{-1}\left[0, \frac{1}{2}\right)$ and $V = f^{-1}\left(\frac{1}{2}, 1\right]$ which satisfy the required conditions.

This completes the proof. ☺

Proof of Urysohn's Lemma

Idea. Given disjoint closed sets A, B:

Step 1. We define a function $f : X \to [0, 1]$ such that

$$f(A) = \{0\} \text{ and } f(B) = \{1\}.$$

Step 2. Then we prove the continuity of f.

Preparation for defining f

Let $P = \mathbb{Q} \cap [0, 1]$.
We construct a collection of open sets

$$\{U_r \mid r \in P\},$$

with the property:

$$p < q \iff \overline{U}_p \subset U_q.$$

We apply the fact below:
Theorem 1. X is normal if and only if for every closed set C in X and an open set $C \subset U \subset X$, there exists a smaller open set V such that

$$C \subset V \subset \overline{V} \subset U.$$

Construction continue. . .

- Let $p < p_{n+1} < q$, where p and q are the immediate successor and predecessor of p_{n+1} respectively.
- We have

$$U_p \subset \underbrace{\overline{U}_p \subset U_q}.$$

- Apply Theorem 1, and get

$$U_p \subset \underbrace{\overline{U}_p \subset U_{n+1} \subset \overline{U}_{n+1} \subset U_q}$$

- Thus we construct $\{U_r \mid r \in P\}$.
- Now, we extend it to all over $\mathbb{Q}$ by defining

$$U_r = \begin{cases} \emptyset & \text{if } r < 0 \\ X & \text{if } r > 1 \end{cases}$$

Definition of f

- Define
$$Q(x) = \{r \in \mathbb{Q} \mid x \in U_r\}.$$
- Now, define $f : X \to [0, 1]$ by
$$f(x) = \inf Q(x) = \inf\{r \in \mathbb{Q} \mid x \in U_r\}.$$
- For $a \in A$, $Q(a) = \{r \in Q \mid r > 0\}$. Therefore
$$f(a) = 0, \ \forall a \in A.$$
- For $b \in B$, $Q(b) = \{r \in \mathbb{Q} \mid r > 1\}$. Therefore,
$$f(b) = 1, \ \forall b \in B.$$

Observations

Ob 1) $x \in \overline{U}_r \implies f(x) \leq r.$
Ob 1) $x \notin U_r \implies f(x) \geq r.$

Continuity of f

- Let $x_0 \in X$ and $(c, d) \subset \mathbb{R}$ such that $f(x_0) \in (c, d)$.
  To show: $\exists\ x_0 \in U \subset_{\text{open}} X$, such that
  $$f(U) \subset (c, d).$$
- Choose rationals p and q such that $c < p < f(x_0) < q < d$, so that $f(x_0) \in (p, q) \subset (c, d)$.
- Consider $U = U_q \setminus \overline{U}_p$.

Construction

- $P = \{p_1 = 1, p_2 = 0, p_3, p_4, \dots\}$. Standard way is:

$$P = \{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}.$$

- Define $U_1 = X \setminus B$.
- $A \subset U_1 \implies \exists V = U_0 \subset_{\text{open}} X$ s.t.

$$A \subset U_0 \subset \overline{U_0} \subset U_1.$$

U_{p_1}, U_{p_2} are defined. We use induction.

- Let U_{p_i}'s are defined for every

$$p_i \in P_i = \{p_1, p_2, \dots, p_n\}.$$

- $P_{n+1} = \{p_1, p_2, \dots, p_{n+1}\}$.

a) $x_0 \in U$:
By Ob 2), $f(x_0) < q \implies x_0 \in U_q$.
By Ob 1) $p < f(x_0) \implies x_0 \notin \overline{U}_p$.
Thus $x_0 \in U$.
b) $f(U) \subset (c, d)$:
Let $x \in U$, then $x \in U_q \subset \overline{U}_q \implies f(x) \leq q$ (using Ob 1))
And $x \notin \overline{U}_p \implies f(x) \geq p$ (using Ob 2)). Thus we have

$$p \leq f(x) \leq q.$$

This completes the proof.


Thank you!